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Bifurcation of Self-Excited Rigid Bodies Subjected to Small Perturbation Torques

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The attitude motion of self-excited rigid bodies subjected to small perturbation torques is investigated by using the version of the Melnikov method developed for slowly varying oscillators. For this purpose the Deprit variables are introduced to transform the equations of motion into a form describing a slowly varying oscillator. For self-excited rigid bodies subjected to small periodic torques, the existence of transversal intersections of heteroclinic orbits has been found for certain parameter ranges. For self-excited rigid bodies subjected to small constant torques, the relationship between the bifurcation (nontransverse) of heteroclinic orbits and the system parameters is given.

Nomenclature

A, B, C	= principal moments of inertia
b	= $\cos^{-1}(L/G)$
D	= $G^2/2\mathcal{H}$
$\mathcal{H}$	= Hamiltonian of the unperturbed system
I	= $\cos^{-1}(H/G)$
L, G, H	= Deprit angular momenta
$\mathcal{L}$	= Lagrangian of the unperturbed system
l, g, h	= Deprit angles
M_x, M_y, M_z	= torque components in the body-fixed coordinate system
$M(t_0), M(\tau_0)$	= Melnikov functions
m_1, m_2, m_3	= constant torque components in the body-fixed coordinate system
p_ψ, p_θ, p_ϕ	= generalized angular momenta conjugate to the Euler angles
$q_0^{\bar{G}}(t)$	= the heteroclinic orbit on the level $G = \bar{G}$
T_1, T_2, T_3	= amplitude components of the periodic torque in the body-fixed coordinate system
W^s, W^u	= stable and unstable manifolds
μ	= friction coefficient
τ	= $\sqrt{[2\mathcal{H}(A-D)(B-C)/ABC]}t$
ψ, θ, ϕ	= Euler angles
$\dot{\psi}, \dot{\theta}, \dot{\phi}$	= Euler angular velocities
Ω	= excitation frequency
$\omega_x, \omega_y, \omega_z$	= angular velocity components in the body-fixed coordinate system

Introduction

THE attitude evolution of a rigid body under various torques has been extensively studied over the past few decades because of its importance in aerospace engineering. Usually, numerical simulations play an important role in understanding the attitude response and control characteristics. On the other hand, analytical models can be of great help in obtaining a qualitative understanding of the complex dynamic behavior of rigid-body motion. Amongst these analytical models, the chaotic attitude motion of satellites in the central gravitational field has been studied by many researchers. The Melnikov method was used to study chaotic and periodic plane motion of asymmetric satellites whose center of mass moves in a prescribed elliptic orbit.[1] Other researchers investigated the chaotic attitude motion of axisymmetric spinning satellites and dual-spin satellites in the gravitational field.[2,3] Chaotic tumbling of Hyperion, one of Saturn's natural satellites, was studied by Wisdom et al.[4] On the other hand, the studies on the attitude motion of the self-excited rigid bodies, that is, a rigid body subjected to a torque that has a fixed direction in the body-fixed coordinate frame, are concentrated more on seeking closed-form solutions analytically. Many classic results about the attitude motion of symmetric or asymmetric self-excited rigid bodies have been summarized by Leimanis.[5] Longuski[6] studied the near-symmetric rigid body under constant body-fixed torques and obtained an analytical solution for the angular velocities and an approximate analytical solution for the Eulerian angles. Van der Ha[7] presented a perturbation solution for the attitude motion of a self-excited rigid body with arbitrary inertia properties under constant body-fixed torques, by taking the ratio of a transverse rotation rate and the spin or scan rate as the small perturbation parameter. Tsiotras and Longuski[8] used the complex variables to express the solution of the near-symmetric self-excited rigid body in a compact form. The accuracy of the obtained solution is tested by numerical simulation of the governing equations. Recently, Longuski and Tsiotras[9,10] considered more complicated cases in which the body-fixed torques vary with time. They derived analytical solutions for the attitude motion of a near-symmetric rigid body subjected to time-varying torques in terms of certain integrals and then proposed a methodology to evaluate these integrals in closed form. However, it is very difficult, if not impossible, to obtain analytical solutions for a self-excited rigid body with arbitrary inertia properties. In this situation, the nonlinear dynamic system theory may provide a tool for the understanding of the complex dynamic behavior for the motion of self-excited rigid bodies. It is well known that, for a torque-free rigid body with distinct moments of inertia, the rotations about the axes of maximum and minimum moments of inertia are stable (center), and the rotation about the axis of medium moment of inertia is unstable (saddle). When a rigid body is subject to a small torque, the heteroclinic orbits (separatrix) that connect the saddles are expected to break and perhaps to intersect transversely. The existence of transversal intersections of heteroclinic (homoclinic) orbits implies complex dynamic behavior in the sense of the Smale horseshoe and provide a necessary condition for chaotic motion to occur. There is an analytical technique, the Melnikov method,[11] to detect the transversal intersections of heteroclinic (homoclinic) orbits and chaotic motion in nonlinear dynamic systems, using ideas that go back to Poincaré.[12] This method has been studied extensively and applied to the study of the chaotic

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motion in physical systems (see, e.g., Refs. 13 and 14 and references therein). For a rigid body subject to small conservative torques, the Melnikov method was extended by Holmes and Marsden[15] for systems with topologically nontrivial phase space. Their version of the Melnikov method was applied to investigate the chaotic motions of a rigid body attached with a flywheel[16] and an asymmetric gyrostat in the gravitational field.[17] Recently, Gray et al.[18,19] used the Melnikov method to detect the chaotic saddles of damped satellites subject to small perturbations due to time-periodic subbody motion in an attitude transition maneuver. In their studies, the spherical coordinates were used to transform the equations of motion into a form suitable for the application of the Melnikov method, and analytical criteria for chaotic motion to occur were derived in terms of the system parameters.

We employ the Melnikov method to investigate the nonlinear attitude motion of asymmetric self-excited rigid bodies subjected to small perturbation torques in a viscous medium. We first introduce the Deprit canonical variables to transform the equations of motion into a form describing a slowly varying oscillator. Then the version of the Melnikov method developed for analyzing the slowly varying oscillator is used to predict the transversal intersections of stable and unstable manifolds for the perturbed self-excited rigid body subjected to small periodic torques. It is shown that there exist transversal intersections of heteroclinic orbits for certain parameter ranges. Thus the motion of self-excited rigid bodies subjected to small periodic torques could become quite complex and may give rise to chaotic motions. On the other hand, the Melnikov method also is used to analyze the attitude motion of self-excited rigid bodies subjected to small constant torques. In this case the relationship between the bifurcation of heteroclinic orbits (nontransverse) and the parameters of the system is obtained. The type of problems we consider is not only of theoretic interest but also of practical importance in satellite design to avoid the occurrence of chaotic motion. For instance, the criterion we obtained would be of great help in designing spinning satellites, where the torques are generated with a fixed direction inside the body as a result of a thruster firing.

Equations of Motion in Terms of Deprit's Canonical Variables

For a rigid body rotating about a fixed point, the Euler equations of motion are

$$A\dot{\omega}_x - (B - C)\omega_y\omega_z = M_x \tag{1}$$

$$B\dot{\omega}_y - (C - A)\omega_z\omega_x = M_y \tag{2}$$

$$C\dot{\omega}_z - (A - B)\omega_x\omega_y = M_z \tag{3}$$

Generally, the acting torque $\boldsymbol{M}$ is a function of the angular velocity $\boldsymbol{\omega}$ and the angular positions of the body-fixed coordinate system with respect to the inertial coordinate system. Thus, for the complete determination of the motion, three additional equations are required. Using the Euler angles (ψ, θ, ϕ), we can write these equations as

$$\dot{\psi} = \frac{\omega_x \sin\phi + \omega_y \cos\phi}{\sin\theta} \tag{4}$$

$$\dot{\theta} = \omega_x \cos\phi - \omega_y \sin\phi \tag{5}$$

$$\dot{\phi} = \omega_z - \frac{(\omega_x \sin\phi + \omega_y \cos\phi)\cos\theta}{\sin\theta} \tag{6}$$

The generalized angular momenta conjugate to the Euler angular velocities $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$ are

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = (A\omega_x \sin\phi + B\omega_y \cos\phi)\sin\theta + C\omega_z \cos\theta \tag{7}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = A\omega_x \cos\phi - B\omega_y \sin\phi \tag{8}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C\omega_z \tag{9}$$

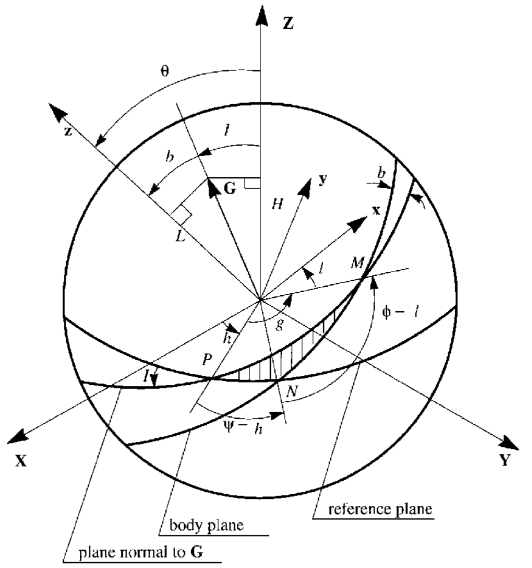

Fig. 1 Euler angles (ψ, θ, ϕ) and the Deprit variables (L, G, H, l, g, h).

where $\mathcal{L} = (A\omega_x^2 + B\omega_y^2 + C\omega_z^2)/2$ is the Lagrangian of the system. Following Deprit,[20] we introduce the following canonical transformation:

$$(p_\psi, p_\theta, p_\phi, \psi, \theta, \phi) \to (L, G, H, l, g, h) \tag{10}$$

where

$$p_\psi = H \tag{11}$$

$$p_\theta = G \sin b \sin(l - \phi) \tag{12}$$

$$p_\phi = L \tag{13}$$

and those angles are related through the following identities of spherical trigonometry applied to the spherical triangle *PNM* (see Fig. 1):

$$\cos I = \cos b \cos\theta + \sin b \sin\theta \cos(\phi - l) \tag{14}$$

$$\cos\theta = \cos I \cos b - \sin I \sin b \cos g \tag{15}$$

$$\sin\theta \cos(\phi - l) = \cos I \sin b + \sin I \cos b \cos g \tag{16}$$

$$\sin\theta \sin(\phi - l) = \sin I \sin g \tag{17}$$

Substituting Eqs. (11–17) into Eqs. (7–9), we can express the angular velocity vector $\boldsymbol{\omega}$ in terms of the Deprit variables as

$$A\omega_x = G \sin b \sin l \tag{18}$$

$$B\omega_y = G \sin b \cos l \tag{19}$$

$$C\omega_z = L \tag{20}$$

where $\cos b = L/G$.

Substitution of Eqs. (18–20) into Eqs. (1–3) yields

$$\dot{G}\sin b \sin l + \dot{b}G \cos b \sin l + \dot{l}G \sin b \cos l - [(B - C)/BC]GL \sin b \cos l = M_x \tag{21}$$

$$\dot{G}\sin b \cos l + \dot{b}G \cos b \cos l - \dot{l}G \sin b \sin l - [(C - A)/CA]GL \sin b \sin l = M_y \tag{22}$$

$$\dot{L} - [(A - B)/(AB)]G^2 \sin^2 b \sin l \cos l = M_z \tag{23}$$

Simplifying these equations, we find

$$\dot{l} = \left(\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)L + \frac{M_x \cos l - M_y \sin l}{G \sin b} \tag{24}$$

$$\dot{L} = [(1/B) - (1/A)](G^2 - L^2)\sin l \cos l + M_z \tag{25}$$

$$\dot{G} = M_z \cos b + (M_x \sin l + M_y \cos l)\sin b \tag{26}$$

For the case of torque-free motion, i.e., $\boldsymbol{M} = 0$, it can be seen from Eq. (26) that the variable G is a constant. The Hamiltonian of the system

$$\mathcal{H} = \frac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right) = \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)(G^2 - L^2) + \frac{L^2}{C} \tag{27}$$

is also a constant. In this case, Eqs. (24) and (25) reduce to

$$\dot{l} = \frac{\partial \mathcal{H}}{\partial L} = \left(\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)L \tag{28}$$

$$\dot{L} = -\frac{\partial \mathcal{H}}{\partial l} = \left(\frac{1}{B} - \frac{1}{A}\right)(G^2 - L^2)\sin l \cos l \tag{29}$$

There exist three equilibrium motions for the torque-free motion in which the rigid body rotates about the three principal axes of moments of inertia, that is,

$$E_1 : l = 0, \pi; \qquad L = 0 \tag{30}$$

$$E_2 : l = \pi/2, 3\pi/2; \qquad L = 0 \tag{31}$$

$$E_3 : |L| = G; \qquad l\text{: arbitrary} \tag{32}$$

The torque-free motion of rigid bodies is one of the few rigid-body problems for which analytical solutions have been found (see, e.g., Ref. 21). Using the Deprit variables, the problem of torque-free motion can be solved more easily than with the conventional manner using the components of the angular velocity vector as basic variables (for details, see Ref. 22). The phase plane of the torque-free motion is constructed in Fig. 2. Without loss of generality, we assume $A > B > C$ in the following discussions. The equilibrium motion E_2 is a center, surrounded by a family of periodic solutions ($G^2/2A < \mathcal{H} < G^2/2B$):

$$\frac{L}{G} = \pm\sqrt{\frac{C(A-D)}{D(A-C)}}\,\mathrm{cn}\,\tau \tag{33}$$

$$\tan l = \mp\sqrt{\frac{A(B-C)}{B(A-C)} + \frac{A(B-D)}{B(A-D)}}\,\frac{\mathrm{dn}\,\tau}{\mathrm{sn}\,\tau} \tag{34}$$

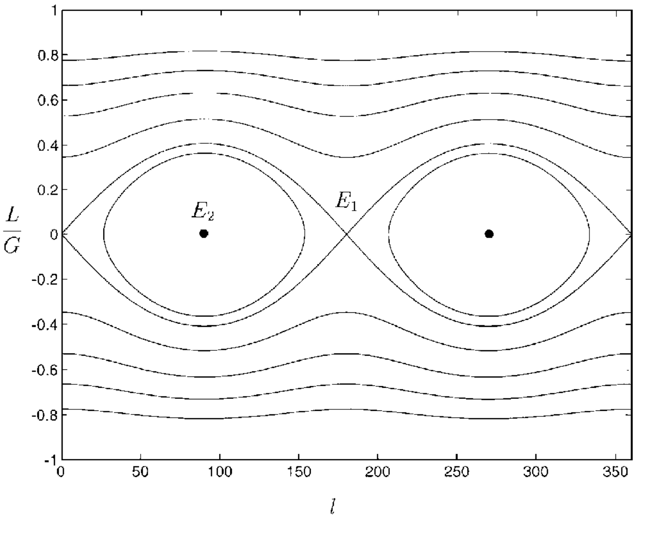

Fig. 2 Phase plane *L/G* vs *l*.

with limits on a pair of heteroclinic orbits $q_0^{\pm}(\tau)$ connecting the unstable equilibrium points E_1. The solutions for the heteroclinic orbits $q_0^{\pm}(\tau)$ are

$$\frac{L}{G} = \pm\sqrt{\frac{C(A-B)}{B(A-C)}}\,\frac{1}{\cosh\tau} \tag{35}$$

$$\tan l = \mp\sqrt{\frac{A(B-C)}{B(A-C)}}\,\frac{1}{\sinh\tau} \tag{36}$$

with the Hamiltonian $\mathcal{H} = G^2/2B$. Outside the heteroclinic orbits, there is another family of periodic solutions ($G^2/2C > \mathcal{H} > G^2/2B$)

$$\frac{L}{G} = \pm\sqrt{\frac{C(A-D)}{D(A-C)}}\,\mathrm{dn}\,\tau \tag{37}$$

$$\tan l = \mp\sqrt{\frac{A(B-C)}{B(A-C)}}\,\frac{\mathrm{cn}\,\tau}{\mathrm{sn}\,\tau} \tag{38}$$

where

$$\tau = \sqrt{\frac{2\mathcal{H}(A-D)(B-C)}{ABC}}\,t, \qquad D = \frac{G^2}{2\mathcal{H}} \tag{39}$$

and $\mathrm{sn}\,\tau$, $\mathrm{cn}\,\tau$, and $\mathrm{dn}\,\tau$ are the Jacobian elliptic functions.

Reduction of Systems to Slowly Varying Oscillators

Because the acting torques do not depend on the orientations of a self-excited rigid body with respect to the inertial coordinate frame, Eqs. (24–26) are independent of the three kinematic equations (4–6) and the angular velocities can be determined through Eqs. (24–26) alone. The perturbation torque may take the form

$$\boldsymbol{M} = \epsilon \begin{bmatrix} -\mu\omega_x + m_1(t) \\ -\mu\omega_y + m_2(t) \\ -\mu\omega_z + m_3(t) \end{bmatrix} \tag{40}$$

where $0 < \epsilon \ll 1$ and $m_1(t)$, $m_2(t)$, $m_3(t)$ are either constants or prescribed functions of time.

Substituting Eq. (40) into Eqs. (24–26), we obtain

$$\dot{l} = \left(\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)L + \epsilon g_l(l, L, G, t) \tag{41}$$

$$\dot{L} = [(1/B) - (1/A)](G^2 - L^2)\sin l \cos l + \epsilon g_L(l, L, G, t) \tag{42}$$

$$\dot{G} = \epsilon g_G(l, L, G, t) \tag{43}$$

where

$$g_l = -\mu\left(\frac{1}{A} - \frac{1}{B}\right)\sin l \cos l + \frac{m_1(t)\cos l - m_2(t)\sin l}{G \sin b} \tag{44}$$

$$g_L = -\mu(L/C) + m_3(t) \tag{45}$$

$$g_G = -\mu\left[\frac{L}{C}\cos b + G\sin^2 b\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)\right] + m_3(t)\cos b + [m_1(t)\sin l + m_2(t)\cos l]\sin b \tag{46}$$

Note that in Eq. (43) the variable G varies slowly in time. This class of system is referred to as slowly varying oscillators,[23,24] which have a common form:

$$\dot{x} = f_1(x, y, z) + \epsilon g_1(x, y, z, t) \tag{47}$$

$$\dot{y} = f_2(x, y, z) + \epsilon g_2(x, y, z, t) \tag{48}$$

$$\dot{z} = \epsilon g_3(x, y, z, t) \tag{49}$$

where $0 < \epsilon \ll 1$. The functions f_1, f_2, and g_i ($i = 1, 2, 3$) are sufficiently smooth (C^r, $r \geq 2$). The Melnikov analytical techniques developed by Wiggins et al.[23,24] for the slowly varying oscillators now can be employed as follows. But first we note that, for $\epsilon = 0$, Eqs. (47–49) reduce to a one-parameter family of planar Hamiltonian systems with Hamiltonian $\mathcal{H}(x, y, z)$:

$$\dot{x} = f_1(x, y, z) = \frac{\partial \mathcal{H}}{\partial y} \tag{50}$$

$$\dot{y} = f_2(x, y, z) = -\frac{\partial \mathcal{H}}{\partial x} \tag{51}$$

$$\dot{z} = 0 \tag{52}$$

Following the method of Wiggins and Shaw,[24] we introduce the following Melnikov integral, which is the first-order (in the perturbation parameter ϵ) measure of the separation of stable and unstable manifolds:

$$M(t_0) = \int_{-\infty}^{\infty} \nabla\mathcal{H}\big[q_0^{z_0}(t)\big] \cdot \boldsymbol{g}\big[q_0^{z_0}(t), t + t_0\big]\mathrm{d}t - \frac{\partial \mathcal{H}}{\partial z}[\gamma(z_0)] \int_{-\infty}^{\infty} g_3\big[q_0^{z_0}(t), t + t_0\big]\mathrm{d}t \tag{53}$$

to determine the existence of transversal intersections of homoclinic orbits. Where $q_0^{z_0}(t)$ is the homoclinic orbit for the unperturbed system (50–52) on $z = z_0$, and

$$\nabla\mathcal{H} = \left(\frac{\partial \mathcal{H}}{\partial x}, \frac{\partial \mathcal{H}}{\partial y}, \frac{\partial \mathcal{H}}{\partial z}\right) \tag{54}$$

while $\boldsymbol{g} = \{g_1, g_2, g_3\}^T$.

For periodic perturbations (g_i are periodic in time), we have the following theorem.

Theorem 1. Suppose there exists a point $\bar{t}_0$ such that

1) $M(\bar{t}_0) = 0$

2) $\dfrac{\partial M}{\partial t_0}(\bar{t}_0) \neq 0$

Then, for ϵ sufficiently small, near $\bar{t}_0$ stable and unstable manifolds intersect transversely.

For the perturbations that do not depend on time explicitly, the Melnikov integral (53) is t_0-independent and transversal intersections of homoclinic orbits do not occur. However, if the perturbation depends on certain parameters of the system, we have Theorem 2.

Theorem 2. Suppose the homoclinic orbit of Eqs. (47–49) depends on a parameter $\mu \in K \subset R$ and such that there exists a point $\mu_0 \in K$ such that

1) $M(\mu_0) = 0$

2) $\dfrac{\partial M}{\partial \mu}(\mu_0) \neq 0$

then for ϵ sufficiently small, near μ_0 there exists a bifurcation value μ at which (nontransverse) homoclinic orbits of (47–49) occur.

For details, see Refs. 23 and 24.

Small Periodic Torques

In this section, we consider the attitude motion of self-excited rigid bodies subjected to a small periodic torque:

$$\boldsymbol{M} = \epsilon \begin{bmatrix} -\mu\omega_x + T_1 \sin\Omega t \\ -\mu\omega_y + T_0 + T_2 \sin\Omega t \\ -\mu\omega_z + T_3 \sin\Omega t \end{bmatrix} \tag{55}$$

where $0 < \epsilon \ll 1$ and $\Omega = \mathcal{O}(1)$.

In contrast to the torque-free motion ($\epsilon = 0$), the stable and unstable manifolds in the perturbed problem ($\epsilon \neq 0$) do not join smoothly. The saddle point γ_0 at $l = 0$ is perturbed as ϵ becomes nonzero, but will become a periodic orbit only under certain conditions. If the perturbed saddle point is periodic, it can be represented by a fixed point on the three-dimensional Poincaré map defined as

$$P: \Sigma \to \Sigma \tag{56}$$

with

$$\Sigma = \{(L, l, G) \mid t = t_n = 2\pi n/\Omega, \quad n = 0, 1, 2, \ldots\} \tag{57}$$

Considering the perturbation on the variable G, we have

$$\dot{G} = -\epsilon[\mu(G/B) - (T_0 + T_2 \sin\Omega t)] + \mathcal{O}(\epsilon^2) \tag{58}$$

The averaged form of Eq. (58) is given by

$$\dot{G} = -\epsilon[\mu(G/B) - T_0] \tag{59}$$

which has a unique fixed point

$$\bar{G} = BT_0/\mu \tag{60}$$

Thus if the above condition is satisfied, i.e., the constant torque is equal to the dissipative torque at $l = 0$, then the perturbed saddle point becomes a periodic orbit. The corresponding fixed point on the Poincaré map is $\gamma_\epsilon(\bar{G}, \epsilon) = \gamma_0(\bar{G}) + \mathcal{O}(\epsilon)$. When $\epsilon = 0$, γ_0 has one stable, one unstable, and one neutral (G) direction. For $0 < \epsilon \ll 1$, the stable and unstable directions for the fixed point γ_ϵ maintain their stability types whereas the neutral direction becomes stable because

$$\frac{\mathrm{d}}{\mathrm{d}G}\left[-\epsilon\left(\mu\frac{G}{B} - T_0\right)\right] = -\epsilon\frac{\mu}{B} < 0 \tag{61}$$

Thus we have

$$\dim\big[W^s(\gamma_\epsilon)\big] = 2, \qquad \dim\big[W^u(\gamma_\epsilon)\big] = 1 \tag{62}$$

The existence of transversal intersections of these stable and unstable manifolds associated with the Poincaré map of the perturbed rigid-body motion now can be determined by computing Melnikov integral (53). Substituting Eqs. (27), (44–46), and (54) into Eq. (53) yields

$$M(t_0) = \int_{-\infty}^{\infty} \left[-\mu L^2\left(\frac{1}{C} - \frac{1}{B}\right)\left(\frac{1}{C} - \frac{1}{A}\right) - T_1\left(\frac{1}{B} - \frac{1}{A}\right)\bar{G}\sin b \sin l \sin\Omega(t + t_0) + T_3\left(\frac{1}{C} - \frac{1}{B}\right)L\sin\Omega(t + t_0)\right]\left(q_0^{\bar{G}}(t)\right)\mathrm{d}t \tag{63}$$

Substituting the solution for the heteroclinic orbits (35) and (36) into Eq. (63) and evaluating the integral by the method of residuals yields

$$M(\tau_0) = -2T_0\sqrt{\frac{(A-B)(B-C)}{AC}} - T_1\sqrt{\frac{C(A-B)}{B(A-C)}} \times \frac{\pi\sin\bar{\Omega}\tau_0}{\cosh(\pi\bar{\Omega}/2)} + T_3\sqrt{\frac{A(B-C)}{B(A-C)}}\frac{\pi\sin\bar{\Omega}\tau_0}{\cosh(\pi\bar{\Omega}/2)} \tag{64}$$

where

$$\bar{\Omega} = \frac{\Omega}{\sqrt{2\mathcal{H}(A-B)(B-C)/ABC}} \tag{65}$$

It can be verified that if

$$-1 \leq \frac{2T_0\sqrt{B(A-B)(B-C)(A-C)}\cosh(\pi\bar{\Omega}/2)}{\pi\sqrt{AC}\left[T_3\sqrt{A(B-C)} - T_1\sqrt{C(A-B)}\right]} \leq 1 \tag{66}$$

then there is a $\bar{\tau}_0$ such that

$$M(\bar{\tau}_0) = 0 \qquad \text{and} \qquad \frac{\partial M}{\partial \tau_0}(\bar{\tau}_0) \neq 0 \tag{67}$$

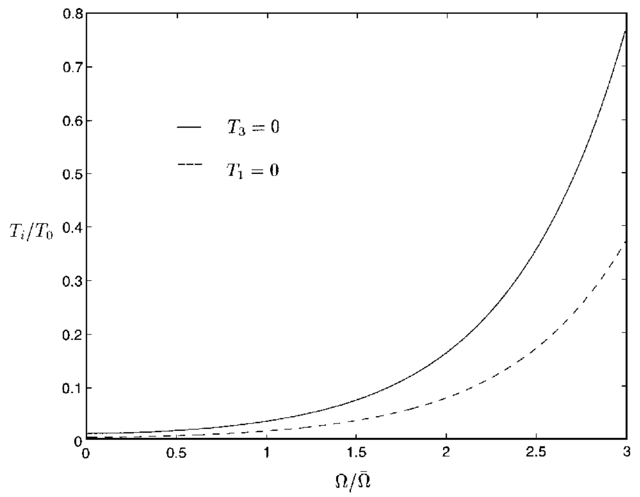

Fig. 3 Bifurcation curves for a homogeneous ellipsoid subjected to a small periodic torque.

Therefore, there exist transversal intersections of stable and unstable manifolds for the perturbed self-excited rigid body.

Let us now consider two special cases. In case 1, the external torque has no components in the $\boldsymbol{e}_z$ direction, i.e., $T_3 = 0$. In this case, the condition for which the transversal intersections of heteroclinic orbits occur is

$$\frac{T_1}{T_0} \geq 2\sqrt{\frac{B(B-C)(A-C)}{AC^2}}\,\frac{\cosh(\pi\bar{\Omega}/2)}{\pi} \tag{68}$$

On the other hand, in case 2, in which there are no external torque components in the $\boldsymbol{e}_x$ direction, i.e., $T_1 = 0$, the condition becomes

$$\frac{T_3}{T_0} \geq 2\sqrt{\frac{B(A-B)(A-C)}{A^2C}}\,\frac{\cosh(\pi\bar{\Omega}/2)}{\pi} \tag{69}$$

Figure 3 shows the bifurcation curves (T_i/T_0 vs $\Omega/\bar{\Omega}$, $i = 1$ or 3) defined by Eqs. (68) and (69) for a homogeneous ellipsoid of unit density, whose three semiaxes are $a = 1$, $b = 2$, and $c = 3$. The corresponding principal moments of inertia are $A = 20.8\pi$, $B = 16\pi$, and $C = 8\pi$.

Small Constant Torques

In this section, we consider the attitude motion of self-excited rigid bodies subjected to a small constant torque

$$\boldsymbol{M} = \epsilon \begin{bmatrix} -\mu\omega_x + m_1 \\ -\mu\omega_y + m_2 \\ -\mu\omega_z + m_3 \end{bmatrix} \tag{70}$$

where $0 < \epsilon \ll 1$ and m_1, m_2, m_3 are constants.

Unlike the periodic case, where the external torque depends on time explicitly, the present system is autonomous and the transversal intersections of heteroclinic orbits due to the perturbation torque cannot be found. However, because the perturbation torque depends on the parameters μ and m_i ($i = 1, 2, 3$), we can use the Melnikov method to find the bifurcation of heteroclinic orbits for this autonomous system and its relationships with these parameters. Substituting Eqs. (27), (44–46), and (54) into (53), we obtain

$$M(\mu) = \int_{-\infty}^{\infty}\left[-\mu L^2\left(\frac{1}{C}-\frac{1}{B}\right)\left(\frac{1}{C}-\frac{1}{A}\right) - m_1\bar{G} \times \left(\frac{1}{B}-\frac{1}{A}\right)\sin b \sin l + m_3 L\left(\frac{1}{C}-\frac{1}{B}\right)\right]\left(q_0^{\bar{G}}(t)\right)\mathrm{d}t \tag{71}$$

where $\bar{G} = Bm_2/\mu$.

Substituting the solution for the heteroclinic orbits (35) and (36) into Eq. (71), upon integration, we obtain the following Melnikov function:

$$M(m_2) = -2m_2\sqrt{\frac{(A-B)(B-C)}{AC}} - m_1\pi\sqrt{\frac{C(A-B)}{B(A-C)}} + m_3\pi\sqrt{\frac{A(B-C)}{B(A-C)}} \tag{72}$$

It can be verified that if

$$m_{2_0} = -\frac{m_1\pi}{2}\sqrt{\frac{AC^2}{B(A-C)(B-C)}} + \frac{m_3\pi}{2}\sqrt{\frac{A^2C}{B(A-B)(A-C)}} \tag{73}$$

then

$$M\left(m_{2_0}\right) = 0 \qquad \text{and} \qquad \frac{\partial M}{\partial m_2}\left(m_{2_0}\right) \neq 0 \tag{74}$$

Therefore, near m_{2_0} the bifurcation (nontransverse) of heteroclinic orbits of the system occurs.

Conclusions

The analog between the motions of self-excited rigid bodies and slowly varying oscillators is revealed by applying the Deprit canonical variables. Thus the nonlinear attitude motion of self-excited rigid bodies under small perturbation torques can be analyzed by using the Melnikov method developed for slowly varying oscillators. For self-excited rigid bodies subjected to small periodic torques, we find the parameter ranges for which there exist transverse intersections of stable and unstable manifolds for the perturbed rigid-body motion. For self-excited rigid bodies subjected to small constant torques, we find the condition under which the bifurcation of heteroclinic orbits occurs.

References

[1]Koch, B. P., and Bruhn, B., "Chaotic and Periodic Motions of Satellites in Elliptic Orbits," *Zeitschrift für Naturforschung*, Vol. 44A, No. 12, 1989, pp. 1155–1162.

[2]Guran, A., Tong, X., and Rimrott, F. P. J., "Instabilities in a Spinning Axisymmetric Rigid Satellite," *Mechanics Research Communications*, Vol. 18, No. 5, 1991, pp. 287–291.

[3]Tong, X., and Rimrott, F. P. J., "Chaotic Attitude Motion of Gyrostat Satellites in a Central Force Field," *Nonlinear Dynamics*, Vol. 4, No. 3, 1993, pp. 269–278.

[4]Wisdom, J., Pearle, S. J., and Mignard, F., "The Chaotic Rotation of Hyperion," *Icarus*, Vol. 58, May 1984, pp. 137–152.

[5]Leimanis, E., *The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point*, Springer–Verlag, New York, 1965.

[6]Longuski, J. M., "On the Attitude Motion of a Self-Excited Rigid Body," *Journal of the Astronautical Sciences*, Vol. 32, No. 4, 1984, pp. 463–473.

[7]Van der Ha, J. C., "Perturbation Solution of Attitude Motion Under Body-Fixed Torques," *Acta Astronautica*, Vol. 12, No. 10, 1985, pp. 861–869.

[8]Tsiotras, P., and Longuski, J. M., "Complex Analytic Solution for the Attitude Motion of a Near-Symmetric Rigid Body Under Body-Fixed Torques," *Celestial Mechanics and Dynamic Astronomy*, Vol. 51, No. 3, 1991, pp. 281–301.

[9]Longuski, J. M., and Tsiotras, P., "Analytic Solution for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torque, Part 1: Constant Axial Torque," *Journal of Applied Mechanics*, Vol. 60, No. 4, 1993, pp. 970–975.

[10]Longuski, J. M., and Tsiotras, P., "Analytic Solution for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torque, Part 2: Time-Varying Axial Torque," *Journal of Applied Mechanics*, Vol. 60, No. 4, 1993, pp. 976–981.

[11]Melnikov, V. K., "On the Stability of the Center for Time Periodic Perturbations," *Transactions of Moscow Mathematical Society*, Vol. 12, No. 1, 1963, pp. 1–57.

[12]Poincaré, H., *Les Methodes Nouvelles de la Mecanique Celeste*, Vol. 3, Gauthier–Villars, Paris, 1899.

[13]Guckenheimer, J., and Holmes, P. J., *Nonlinear Oscillators, Dynamical Systems, and Bifurcations of Vector Fields*, 2nd ed., Springer–Verlag, New York, 1985.

[14]Wiggins, S., *Global Bifurcations and Chaos*, Springer–Verlag, New York, 1988.

[15]Holmes, P. J., and Marsden, J. E., "Horseshoes and Arnold Diffusion for Hamiltonian Systems on Lie Group," *Indiana University Mathematics Journal*, Vol. 32, No. 2, 1983, pp. 273–309.

[16] Koiller, J., “A Mechanical System with a ‘Wild’ Horseshoe,” *Journal of Mathematical Physics*, Vol. 25, No. 5, 1984, pp. 1599–1604.

[17] Tong, X., Tabarrok, B., and Rimrott, F. P. J., “Chaotic Motion of an Asymmetric Gyrostat in the Gravitational Field,” *International Journal of Non-Linear Mechanics*, Vol. 30, No. 3, 1995, pp. 191–203.

[18] Gray, G. L., Kammer, D. C., and Dobson, I., “Chaos in an Attitude Maneuver of a Damped Satellite to Time-Periodic Perturbations,” *Proceedings of the 1992 AAS/AIAA Spaceflight Mechanics Meeting* (Colorado Springs, CO), Vol. 79, Pt. 1–2, Advances in the Astronautical Sciences, Univelt, San Diego, CA, 1992, pp. 593–612 (AAS Paper 92-171).

[19] Gray, G. L., Dobson, I., and Kammer, D. C., “Detection of Chaotic Saddles in an Attitude Maneuver of a Spacecraft Containing a Viscous Damper,” *Proceedings of the 1993 AAS/AIAA Spacecraft Mechanics Meeting* (Pasadena, CA), Vol. 82, Pt. 1–2, Advances in the Astronautical Sciences, Univelt, San Diego, CA, 1993, pp. 167–184 (AAS Paper 93-133).

[20] Deprit, A., “Free Rotation of a Rigid Body Studied in the Phase Plane,” *American Journal of Physics*, Vol. 35, No. 5, 1967, pp. 424–428.

[21] Rimrott, F. P. J., *Introductory Attitude Dynamics*, Springer–Verlag, New York, 1988.

[22] Tong, X., and Tabarrok, B., “Melnikov’s Method for Rigid Bodies Subject to Small Perturbation Torques,” *Archive of Applied Mechanics*, Vol. 66, No. 4, 1996, pp. 215–230.

[23] Wiggins, S., and Holmes, P. J., “Homoclinic Orbits in Slowly Varying Oscillators,” *SIAM Journal on Mathematical Analysis*, Vol. 18, May 1987, pp. 612–629.

[24] Wiggins, S., and Shaw, S. W., “Chaos and Three-Dimensional Horseshoe in Slowly Varying Oscillators,” *Journal of Applied Mechanics*, Vol. 55, No. 4, 1988, pp. 959–968.